Solution to Homework 5 problem 2.

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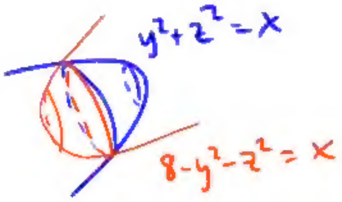
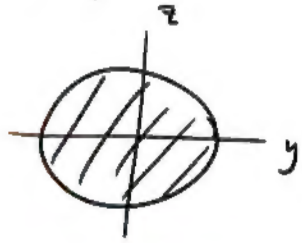
Problem: Set up (but do not evaluate) an iterated integral for the volume of the region R bounded by $x=y^2+z^2$ and $x=8-y^2-z^2$. Make one integral each for Cartesian, Cylindrical, and spherical coordinates.

Sketch:



Cartesian: The shadow of this region in the yz-plane is a circular disk. It is controlled by the intersection of the two paraboloids: $y^2+z^2 = 8-y^2-z^2$ yields $y^2+z^2=4$.

$\therefore$ The shadow is a disk of radius 2.



The shadow is parametrized by

$$\{(x,y) : -2 \le y \le 2,\ -\sqrt{4-y^2} \le z \le \sqrt{4-y^2}\}$$

Solved $y^2+z^2 \le 4$.

Above each point (x,y) in this disk we have possible x-values $y^2+z^2 \le x \le 8-y^2-z^2$. Hence we have parametrized

$$R = \{(x,y,z) : -2 \le y \le 2,\ -\sqrt{4-y^2} \le z \le \sqrt{4-y^2},\ y^2+z^2 \le x \le 8-y^2-z^2\}.$$

$$\therefore \quad \mathrm{Vol}(R) = \iiint_R 1\, dV = \int_{y=-2}^{2} \int_{z=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x=y^2+z^2}^{8-y^2-z^2} dx\, dz\, dy.$$

Cylindrical: We know from above that the yz-shadow of R is a disk, so we shall use coordinates $\begin{cases} x = x \\ y = r\cos(\theta) \\ z = r\sin(\theta) \end{cases}$.

Our yz-shadow is then parameterized by

$$\{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

simply because it is a full disk of radius 2 centered at the origin in the yz-plane.

Above a point (r,θ) we have possible x-values

$$r^2 = (r\cos(\theta))^2 + (r\sin(\theta))^2 \le x \le 8 - (r\cos(\theta))^2 - (r\sin(\theta))^2 = 8 - r^2.$$

Hence we have reparameterized the region

$$R_{cyl} = \{(r,\theta,x) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ r^2 \le x \le 8 - r^2\}$$

and we may thus write our volume integral as follows

$$\begin{aligned} \mathrm{Vol}(R) &= \iiint_{R_{cart}} 1 \, dV_{cart} \\ &= \iiint_{R_{cyl}} 1 \cdot r \, dV_{cyl} \\ &= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \int_{x=r^2}^{8-r^2} r \, dx \, d\theta \, dr \end{aligned}$$

Jacobian of the cylindrical transformation

If you got this far, you should email me with subject line "Somebody reads the handouts" for a full quiz mark.

Spherical: To reparameterize R in spherical coordinates, we use spherical transformation $\begin{cases} x = \rho\cos(\varphi) \\ y = \rho\sin(\varphi)\cos(\theta) \\ z = \rho\sin(\varphi)\sin(\theta) \end{cases}$ because the yz-shadow of the region is a disk (see above), and thus our region still needs a full $0 \le \theta \le 2\pi$. Note that our ρ-bounds depend on φ: the angle φ_0 at the intersection of the two paraboloids marks the changeover point where the outer ρ-bound is determined by $x = 8 - y^2 - z^2$ and then to $x = y^2 + z^2$. However, as a whole we have $0 \le \varphi \le \frac{\pi}{2}$ as every cone above the yz-plane intersects the region R. (the last two sentences mean: the upper bound $\rho_{up}(\varphi)$ is a piecewise function with split-point φ_0)

Now φ_0 is determined by $8 - y^2 - z^2 = y^2 + z^2$, i.e. $y^2 + z^2 = 4$ at the intersection of the upper and lower paraboloids.

As $x = y^2 + z^2 = 4$, we have $\rho_0\cos(\varphi_0) = \underbrace{\rho_0^2\sin^2(\varphi_0)}_{y^2+z^2} = 4$, so $\rho_0\sin(\varphi_0) = 2$ (reject negative root b/c $\rho_0 \ge 0$ and $0 \le \varphi_0 \le \pi$ yields $0 \le \sin(\varphi_0) \le 1$) and we thus have $\tan(\varphi_0) = \frac{\rho_0\sin(\varphi_0)}{\rho_0\cos(\varphi_0)} = \frac{2}{4} = 1/2$. In particular we have $\varphi_0 = \arctan(\frac{1}{2})$. We now find the upper ρ-bounds by cases on φ (b/c it's piecewise-defined).

If $0 \le \varphi \le \arctan(\frac{1}{2})$, then the upper bound for ρ is determined by $x = 8 - y^2 - z^2$, i.e. $\rho\cos(\varphi) = 8 - \rho^2\sin^2(\varphi)$.

Thus $\sin^2(\varphi)\rho^2 + \cos(\varphi)\rho - 8 = 0$ yields by the quadratic formula

$$\rho = \frac{-\cos(\varphi) \pm \sqrt{\cos^2(\varphi) - 4\sin^2(\varphi)(-8)}}{2\sin^2(\varphi)} = \frac{-\cos(\varphi) \pm \sqrt{1 + 31\sin^2(\varphi)}}{2\sin^2(\varphi)}.$$

Rejecting the negative (b/c $\rho \ge 0$) we have $0 \le \rho \le \frac{\sqrt{1+31\sin^2(\varphi)} - \cos(\varphi)}{2\sin^2(\varphi)}$.

If $\arctan(\frac{1}{2}) \le \varphi \le \pi/2$, then the upper bound for ρ is given by $x = y^2 + z^2$, i.e. $\rho\cos(\varphi) = \rho^2\sin^2(\varphi)$.

Thus, we see that away from the origin (i.e. when $\rho \ne 0$) we have $0 \le \rho \le \frac{\cos(\varphi)}{\sin^2(\varphi)} = \csc(\varphi)\cot(\varphi)$ (Note that this bounding function is continuous on $0 < \varphi < \pi$, so this remains a proper integral).

Hence we have reparameterized the region R in two pieces

$$R_{sph} = R_{out} \cup R_{in} \quad \text{where:}$$

$$R_{out} = \left\{ (\rho,\theta,\varphi) : 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \arctan(\tfrac{1}{2}),\ 0 \le \rho \le \frac{\sqrt{1+31\sin^2(\varphi)} - \cos(\varphi)}{2\sin^2(\varphi)} \right\}$$

$$R_{in} = \left\{ (\rho,\theta,\varphi) : 0 \le \theta \le 2\pi,\ \arctan(\tfrac{1}{2}) \le \varphi \le \tfrac{\pi}{2},\ 0 \le \rho \le \csc(\varphi)\cot(\varphi) \right\}$$

Thus we may write our volume as an iterated integral via

$$\mathrm{Vol}(R) = \iiint_{R_{cart}} 1 \, dV_{cart}$$

Jacobian of the spherical transformation

$$= \iiint_{R_{sph}} 1 \cdot \rho^2\sin(\varphi)\, dV_{sph}$$

$$= \iiint_{R_{out} \cup R_{in}} \rho^2\sin(\varphi)\, dV_{sph}$$

$$= \iiint_{R_{out}} \rho^2\sin(\varphi)\, dV_{sph} + \iiint_{R_{in}} \rho^2\sin(\varphi)\, dV_{sph}$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\arctan(\frac{1}{2})} \int_{\rho=0}^{\frac{\sqrt{1+31\sin^2(\varphi)} - \cos(\varphi)}{2\sin^2(\varphi)}} \rho^2\sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$+ \int_{\theta=0}^{2\pi} \int_{\varphi=\arctan(\frac{1}{2})}^{\pi/2} \int_{\rho=0}^{\csc(\varphi)\cot(\varphi)} \rho^2\sin(\varphi)\, d\rho\, d\varphi\, d\theta$$